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OVSYANNIKOV PLANE VORTEX: THE EQUATIONS OF THE SUBMODEL

S. V. Golovin

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A submodel of the equations of ideal magnetohydrodynamics is constructed that generalizes the classical motion of an ideal continuous medium with plane waves. It is shown that, in contrast to classical motion, in this submodel the velocity and magnetic-field vectors can change direction in a plane orthogonal to a distinguished spatial direction. The submodel is described by a system of equations with two independent variables and a finite relation specifying the orientation of the vector fields in space. The solutions of the submodel define substantially spatial processes and singularities in the motion of continuous media which cannot be studied in the classical one-dimensional formulation.

Key words: *ideal magnetohydrodynamics, partially invariant solutions, overdetermined systems of differential equations, singularities in the motion of a continuous medium.*

INTRODUCTION

In the classical one-dimensional motion of an ideal continuous medium, functions depend only on two variables — time t and the Cartesian spatial coordinate x. Particle motion is admitted in all directions (the velocity vector has three components), but the main perturbations (compression and rarefaction waves, strong and weak discontinuities, etc.) occur only along the spatial Ox axis. The velocity components and thermodynamic functions in the planes $x = \text{const}$ are constant and can vary only from one plane to another. This fairly simple solution, called plane-wave motion, contains a large body of information on the motion of continuous media. However, this simplification is unsuitable in the case of substantially three-dimensional processes in fluids, which can be important for a correct description of the motion.

In the present work, it is proposed to generalize the classical solution as follows. The velocity vector is split into two components: a component parallel to the Ox axis and a component orthogonal to it. It is assumed that the lengths of both components and thermodynamic functions depend only on the variables t and x (which corresponds to the classical solution). The generalization of the classical solution consists of the fact that the angle of rotation of the particle velocity around the Ox axis is assumed to depend on all independent variables.

The examined generalization of the classical one-dimensional motion follows from the symmetry properties of mathematical models [1, 2]. In terms of the group analysis of differential equations, the classical solution is invariant under the group of translations along the Oy and Oz axes whereas the generalized solution is partially invariant under group of plane isometries (translations along the Oy and Oz axes, and rotations around the Ox axis). This class of solutions for the equations of an ideal continuous medium is substantial and is described by a closed invariant system of equations with two independent variables, which, in a particular case, leads to the classical equations of one-dimensional plane-wave motion. The no invariant function is determined for the solutions of the invariant system from a finite (nondifferential) relation containing a functional arbitrariness. For this relation there is a clear geometrical interpretation which allows one to construct motion of the required type by selecting the arbitrary function included in the relation.

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; sergey@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 49, No. 5, pp. 27–40, September–October, 2008. Original article submitted July 30, 2007; revision submitted September 5, 2007.

It should be noted that the solution of the equations of an ideal continuous medium called the singular vortex or the Ovsyannikov vortex [3–11], which has now been extensively studied, is a similar generalization of the classical one-dimensional motion with spherical waves. In this solution, the moduli of the velocity components which are normal and tangent to spheres $r = \text{const}$ are invariant quantities, i.e., they depend only on time t and the distance to the coordinate origin r, whereas the angle of rotation of the velocity vector around the radial direction Or is a noninvariant quantity and depends on all independent variables. This solution also has a group nature, i.e., it is partially invariant under the admitted group of sphere isometries (rotations in three-dimensional space).

The examined solution for the equations of ideal gas dynamics was first obtained in [12], but it has not been analyzed in detail from the point of view of the physical content. In the present work, we study the equations of ideal magnetohydrodynamics, which, for zero magnetic field ($\boldsymbol{H} = 0$) and constant density ($\rho = \text{const}$), lead to generalized one-dimensional motion for ideal fluid dynamics. A method for obtaining the equations of the submodel and a geometrical interpretation of the implicit finite relations for the noninvariant function included in the solution are discussed. The physical properties of the motion of a continuous medium described by the solution are studied in [13].

1. CONSTRUCTION OF THE SUBMODEL

1.1. Representation of the Solution. The initial system of equations of ideal magnetohydrodynamics (the stress is reduced to pressure, zero thermal conductivity, infinite electrical conductivity) is written as follows [14]:

$$D\rho + \rho \operatorname{div} \boldsymbol{u} = 0; \tag{1}$$

$$D\boldsymbol{u} + \rho^{-1}\nabla p + \rho^{-1}\boldsymbol{H} \times \operatorname{rot} \boldsymbol{H} = 0; \tag{2}$$

$$Dp + A(p, \rho) \operatorname{div} \boldsymbol{u} = 0; \tag{3}$$

$$D\boldsymbol{H} + \boldsymbol{H} \operatorname{div} \boldsymbol{u} - (\boldsymbol{H} \cdot \nabla)\boldsymbol{u} = 0; \tag{4}$$

$$\operatorname{div} \boldsymbol{H} = 0, \qquad D = \partial_t + \boldsymbol{u} \cdot \nabla, \tag{5}$$

where $\boldsymbol{u} = (u, v, w)$ is the velocity vector, $\boldsymbol{H} = (H, K, L)$ is the magnetic-field intensity vector, and p and ρ are the pressure and density. The equation of state $p = F(S, \rho)$ with entropy S is valid. The function $A(p, \rho)$ is given by the equation of state $A = \rho\,(\partial F/\partial \rho)$. All functions depend on time t and the Cartesian coordinates $\boldsymbol{x} = (x, y, z)$.

For an arbitrary equation of state $p = F(S, \rho)$, Eqs. (1)–(5) admit a 11-dimensional Lie group G_{11} of point transformations, which is an extension of the 10-dimensional Galilean group to the homothety transformation [15, 16]. The optimal system of subgroups ΘG_{11} is constructed in [17, 18], and in the final form in [19]. An analysis of the system ΘG_{11} shows that the partially invariant solution described here is generated by the subgroup $G_{3.13} \subset G_{11}$ with the Lie algebra $L_{3.13}$ of infinitesimal generators $\{\partial_y,\ \partial_z,\ z\,\partial_y - y\,\partial_z + w\,\partial_v - v\,\partial_w + L\,\partial_K - K\,\partial_L\}$ (the numbering of subalgebras is the same as in [19]).

Indeed, the group $G_{3.13}$ is generated by shifts along the Oy and Oz axes with simultaneous rotation around the first coordinate axis in the spaces $\mathbb{R}^3(\boldsymbol{x})$, $\mathbb{R}^3(\boldsymbol{u})$, and $\mathbb{R}^3(\boldsymbol{H})$. In the space $\mathbb{R}^4(t, \boldsymbol{x}) \times \mathbb{R}^8(\boldsymbol{u}, \boldsymbol{H}, p, \rho)$, the invariants of this group of transformations have the form

$$t, \quad x, \quad u, \quad V = \sqrt{v^2 + w^2}, \quad p, \quad \rho, \quad H, \quad N = \sqrt{K^2 + L^2}, \quad vK + wL. \tag{6}$$

The last invariant can be interpreted as the angle σ between the projections of the vectors $\boldsymbol{u}$ and $\boldsymbol{H}$ onto the plane Oyz (Fig. 1). General theory for partially invariant solutions is presented in [1]. A representation for the partially invariant solution can be obtained by specifying functional dependences between the invariants (6). In particular, for the solution of rank 2 (two invariant independent variables) and defect 1 (one noninvariant function) we obtain the following representation of the solution:

$$\begin{aligned} &u = U(t, x), && H = H(t, x), \\ &v = V(t, x)\cos\omega(t, x, y, z), && K = N(t, x)\cos(\omega(t, x, y, z) + \sigma(t, x)), \\ &w = V(t, x)\sin\omega(t, x, y, z), && L = N(t, x)\sin(\omega(t, x, y, z) + \sigma(t, x)), \end{aligned} \tag{7}$$
$$p = p(t, x), \qquad \rho = \rho(t, x), \qquad S = S(t, x).$$

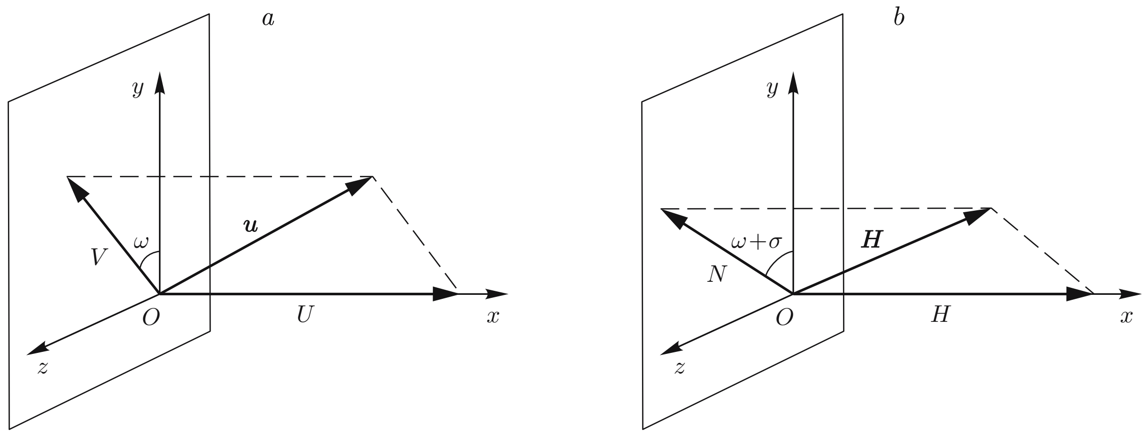

Fig. 1. Representations of the velocity vectors $\boldsymbol{u}$ (a) and magnetic-field intensity $\boldsymbol{H}$ (b) in the partially invariant solution.

The unique noninvariant function $\omega(t, x, y, z)$ (see Fig. 1) depends on all initial independent variables. The invariant functions U, V, H, N, σ, p, and ρ are assumed to be dependent on the invariant variables t and x. The system of equations for the invariant and noninvariant functions will be called the submodel of the initial model of ideal magnetohydrodynamics.

1.2. Preliminary Analysis. Substitution of representation (7) into the continuity equation (1) allows one to introduce a new invariant unknown function $h(t, r)$ determined from the relation

$$\tilde{D}\rho + \rho(U_x + hV) = 0. \tag{8}$$

Here and below, the operator $\tilde{D}$ denotes the invariant part of the operator of differentiation along the trajectory:

$$\tilde{D} = \partial_t + U\partial_x.$$

The remaining part of the continuity equation (1) leads to the first equation for the noninvariant function ω:

$$\sin\omega\,\omega_y - \cos\omega\,\omega_z + h = 0. \tag{9}$$

In addition, from the equations for the invariant functions, we obtain the first components of the momentum equation (2), the induction equation (4), and the equation for pressure (3):

$$\tilde{D}U + \rho^{-1}p_x + \rho^{-1}NN_x = 0; \tag{10}$$

$$\tilde{D}H + hHV = 0; \tag{11}$$

$$\tilde{D}p + A(p, \rho)(U_x + hV) = 0. \tag{12}$$

The remaining five equations of system (1)–(5) lead to an overdetermined system for the function ω. Forming nondegenerate linear combinations of Eqs. (2) in projections onto the Oy and Oz axis, we have

$$\begin{gathered}\rho V\omega_t + \Big(\rho UV - HN\cos\sigma\Big)\,\omega_x + \Big(\rho V^2\cos\omega - N^2\cos\sigma\cos{(\omega+\sigma)}\Big)\,\omega_y \\ + \Big(\rho V^2\sin\omega - N^2\cos\sigma\sin{(\omega+\sigma)}\Big)\,\omega_z - H(N_x\sin\sigma + N\cos\sigma\sigma_x) = 0;\end{gathered} \tag{13}$$

$$\begin{gathered}HN\sin\sigma\,\omega_x + N^2\sin\sigma\cos{(\omega+\sigma)}\,\omega_y + N^2\sin\sigma\sin{(\omega+\sigma)}\,\omega_z \\ + \rho\,\tilde{D}V + HN\sin\sigma\,\sigma_x - HN_x\cos\sigma = 0.\end{gathered} \tag{14}$$

Similar manipulations with the remaining two induction equations (4) yield

$$\begin{gathered}N\omega_t + (NU - HV\cos\sigma)\,\omega_x + VN\sin\sigma\sin{(\omega+\sigma)}\,\omega_y \\ - VN\sin\sigma\cos{(\omega+\sigma)}\,\omega_z + N\tilde{D}\sigma + HV_x\sin\sigma = 0;\end{gathered} \tag{15}$$

$$HV\sin\sigma\,\omega_x + NV\cos\sigma\sin(\omega+\sigma)\,\omega_y - NV\cos\sigma\cos(\omega+\sigma)\,\omega_z - \tilde{D}N + HV_x\cos\sigma - NU_x = 0. \tag{16}$$

Equation (5) is written as

$$N\Big(\sin(\omega+\sigma)\omega_y - \cos(\omega+\sigma)\,\omega_z\Big) - H_x = 0. \tag{17}$$

We examine the compatibility of system (9), (13)–(17) for the noninvariant function ω, seeking solutions in which the function ω is determined with a functional arbitrariness. According to the reduction theorem [1], if this condition is not satisfied, the solution reduces to an invariant solution with respect to a certain two-dimensional subalgebra of the initial algebra $L_{3.13}$. In the algebra $L_{3.13}$ there is only one two-dimensional subalgebra $\{\partial_y, \partial_z\}$. The corresponding invariant solution is the classical one-dimensional motion of plasma. Thus, the discarded solution is already known.

1.3. Prohibition of Reduction. To prohibit reduction of the solution, it is necessary to rule out the situation in which all derivatives of the noninvariant function ω can be expressed from system (9), (13)–(17). For this, it is necessary to calculate the coefficient matrix for the derivatives of the function ω and to equate the rank minors to zero. As in the case of the spherical singular vortex [9], this is possible only if one of the following conditions are satisfied:

$$\begin{aligned} &1)\quad H = 0;\\ &2)\quad N = 0;\\ &3)\quad V = 0;\\ &4)\quad \sigma = 0 \text{ or } \sigma = \pi. \end{aligned} \tag{18}$$

In this case, we assume that the inequalities $H^2 + N^2 \neq 0$ and $U^2 + V^2 \neq 0$ are satisfied (in the case of no magnetic field or in the state of magnetohydrodynamic equilibrium, the solution is irreducible). For case 4, we note that, by definition (7), the functions V and N are nonnegative. However, the values $\sigma = \pi$ and $\sigma = 0$ differ only in the sign of the function N; therefore, in the following consideration of case 4, we shall assume that the function V is nonnegative and the function N can have an arbitrary sign.

According to classification (18), variants 2 and 3 correspond to a magnetic field parallel to the Ox axis and the plasma velocity field. As in the case of the Ovsyannikov spherical vortex, these variants are contained in variant 4 ($\sigma = 0$). Indeed, the equality $\sigma = 0$ implies that the velocity vector and the magnetic-field intensity vector of each particle are in one plane which is orthogonal to the coordinate plane Oyz. Then, the cases where the magnetic field or the velocity field are parallel to the plane Oyz are degenerate variants of this more the general situation. Below, case 4 will be considered as the basic one.

We note that the condition of irreducibility of the solution imposes a constraint on the arrangement of the velocity and magnetic field vectors. Generally speaking, in classical one-dimensional plane-wave motion, such constraints are absent. This implies that in the case of magnetohydrodynamics, solution (7) extends the classical one-dimensional plane-wave solution only to the class of motions for which one of conditions (18) is satisfied. This constraint of the class of solutions is due to the fact that the employed condition of reducibility of solutions is only a sufficient one. This implies that different irreducible solutions can exist, which are not contained in the classification (18) and completely fill the missing part of the extension of the classical solution. However, this missing part can be trivial, i.e., if conditions (18) are satisfied, the unique lost solution is the classical one-dimensional solution. Therefore, it is of interest to study the overdetermined system (9), (13)–(17) without using the assumption of reduction prohibition. The above remarks refer only to the equations of magnetohydrodynamics since the latter contain two vector fields — of the velocity and magnetic-field intensity. For the case of pure gas dynamics $\boldsymbol{H} \equiv 0$, three of the four conditions (18) are satisfied automatically; therefore the irreducibility property does not impose additional constraints on the solution.

2. ANALYSIS OF THE SOLUTION

2.1. Case of a Plane Magnetic Field. We consider the case $H = 0$, where the magnetic-field vector is parallel to the plane Oyz. In this case, the compatibility condition for Eqs. (9) and (17) is written as

$$(\cos(\omega+\sigma)\,\omega_y + \sin(\omega+\sigma)\,\omega_z)h = 0. \tag{19}$$

In the case $h = 0$, the determinant of the homogeneous system (9), (17) for ω_y and ω_z is $\sin\sigma$, i.e., the solution is nontrivial if $\sigma = 0$ or $\sigma = \pi$. The case $h \neq 0$ leads to reduction, as follows from Eqs. (17) and (19). Thus, a nontrivial solution exists only for $\sigma = 0$, i.e., according to classification (18), case 1 is contained in case 4.

Assuming that $N \neq 0$, we study in greater detail the equations of the obtained submodel for $H = h = \sigma = 0$. Equation (9) becomes homogeneous and has the form

$$\sin\omega\,\omega_y - \cos\omega\,\omega_z = 0. \tag{20}$$

Equation (13) leads to

$$\rho V\,\omega_t + \rho UV\,\omega_x + (\rho V^2 - N^2)(\cos\omega\,\omega_y + \sin\omega\,\omega_z) = 0. \tag{21}$$

Using (20), from Eq. (14), we obtain

$$\tilde{D}V = 0. \tag{22}$$

Dividing (15) by the nonzero multiplier N, we have

$$\tilde{D}\omega = 0. \tag{23}$$

Simplifying Eq. (16) with the use of (20), we obtain

$$\tilde{D}N + NU_x = 0.$$

Finally, Eq. (17) coincides with (20). Thus, in case of a magnetic field parallel to the plane Oyz, the examined class of plasma motions is described by the following system of equations for the invariant functions:

$$\begin{gathered}\tilde{D}U + \rho^{-1}p_x + \rho^{-1}NN_x = 0, \qquad \tilde{D}p + A(p,\rho)U_x = 0,\\ \tilde{D}\rho + \rho U_x = 0, \qquad \tilde{D}N + NU_x = 0, \qquad \tilde{D}V = 0.\end{gathered} \tag{24}$$

The last equation of system (24) is independent and can be solved separately. An integral of the equation for N is the function $N = N(\rho)$. Introducing the new functions $P = p + (1/2)N(\rho)^2$ and $A_1 = A(p,\rho) + N(\rho)N'(\rho)$, we reduce Eqs. (24) to the equations of one-dimensional gas dynamics:

$$\tilde{D}U + \rho^{-1}P_x = 0, \qquad \tilde{D}p + A_1(p,\rho)U_x = 0, \qquad \tilde{D}\rho + \rho U_x = 0.$$

For the noninvariant function ω, we obtain system (20)–(23), which, for $\rho V^2 - N^2 \neq 0$, implies that $\omega_y = \omega_z = 0$, i.e., the solution is reduced to classical one-dimensional plane-wave motion. In the case $\rho V^2 - N^2 = 0$, in view of $N = N(\rho)$, we obtain the function $V = V(\rho)$, from which it follows that $\tilde{D}\rho = 0$, and, hence, $U_x = 0$. Consequently, the functions V, N, p, and ρ are arbitrary functions of the variable $x - \displaystyle\int U(t)\,dt$. The first equation of system (24) implies that $U'(t) = \text{const}$. Thus, the following theorem is valid:

Theorem 1. *In the case of a plane magnetic field $H = 0$, the examined class of solutions reduces to the classical one-dimensional plane-wave solution or describes uniformly accelerated plasma motion in the Ox direction. In the latter case, all required functions are constant in planes orthogonal to the Ox axis and moving at the same acceleration along this axis.*

2.2. Basic Case. We consider the case $\sigma = 0$. From a physical point of view, this case is equivalent to plasma flow in which the velocity and magnetic-field vectors belong to a plane parallel to the directing vector of the Ox axis. The invariant part of the solution is given by Eqs. (8) and (10)–(12). In addition, in view of (9), Eq. (14) leads to

$$\tilde{D}V - \rho^{-1}HN_x = 0.$$

From equation (16), we obtain

$$\tilde{D}N + NU_x - HV_x + hNV = 0,$$

and Eq. (17) is written as

$$H_x + hN = 0. \tag{25}$$

The noninvariant part of the solution is defined, in addition to (9), by the equations following from (13) and (15):

$$\rho V\,\omega_t + (\rho UV - HN)\,\omega_x + (\rho V^2 - N^2)(\cos\omega\,\omega_y + \sin\omega\,\omega_z) - HN\sigma_x = 0; \tag{26}$$

$$N\,\omega_t + (NU - HV)\,\omega_x = 0. \tag{27}$$

Eliminating the derivative ω_t from Eqs. (26) and (27), we obtain the classifying relation

$$(\rho V^2 - N^2)\Big(H\omega_x + N(\cos\omega\,\omega_y + \sin\omega\,\omega_z)\Big) = 0. \tag{28}$$

Let us consider the basic case where the second multiplier in relation (28) is equal to zero. The compatibility conditions for Eqs. (9), (27), and (28) are written as

$$N\tilde{D}h - HVh_x = 0, \qquad Hh_x + h^2N = 0.$$

Taking into account that, for $h \neq 0$, there exists the integral

$$H = H_0 h, \tag{29}$$

the equations of the submodel can be reduced to the equations

$$\tilde{D}\rho + \rho(U_x + hV) = 0; \tag{30}$$

$$\tilde{D}U + \rho^{-1}p_x + \rho^{-1}NN_x = 0; \tag{31}$$

$$\tilde{D}V - \rho^{-1}H_0hN_x = 0; \tag{32}$$

$$\tilde{D}p + A(p,\rho)(U_x + hV) = 0; \tag{33}$$

$$\tilde{D}N + NU_x - H_0hV_x + hNV = 0; \tag{34}$$

$$\tilde{D}h + Vh^2 = 0, \qquad H_0h_x + hN = 0. \tag{35}$$

System (30)–(35) is overdetermined [similarly to the initial system of magnetohydrodynamics (1)–(5)]: it contains seven equations for six required functions and is in involution. Indeed, the last two equations for the function h imply the unique nontrivial compatibility condition for this system. Performing cross differentiation of Eqs. (35), as the compatibility condition we obtain Eq. (34), which is already contained in the system. To formulate the Cauchy problem for system (30)–(35), it is necessary to specify the functions ρ, U, V, p, and N of x at $t = 0$ and to fix a constant value of the function h in a certain plane $x = \text{const}$. In the case of ideal gas dynamics $\boldsymbol{H} \equiv 0$, overdetermination is absent. The second equation in (35) is satisfied identically; therefore, the initial data for h are specified in the same manner as for the remaining functions: $h(0,x) = h_0(x)$. System (30)–(35) with the initial data can be studied numerically or analytically or by using methods of group analysis of differential equations since the group admitted by this system is nontrivial.

The equations for the noninvariant function are integrated, and the function ω is determined from the implicit relation

$$F(y - \tau\cos\omega, z - \tau\sin\omega) = 0, \tag{36}$$

where $\tau = 1/h$ and F is an arbitrary smooth function. In the case of ideal gas dynamics $\boldsymbol{H} \equiv 0$, Eq. (27) is satisfied identically; therefore, in (36) the function F also depends arbitrarily on ξ: $\xi_t + U\xi_x = 0$. The result obtained is formulated in the following theorem.

Theorem 2. *In the basic case $h \neq 0$, the invariant functions are found from the system of differential equations* (30)–(35). *The noninvariant function ω is given by the finite relation* (36) *with an arbitrary smooth function F.*

2.3. Geometrical Construction of the Direction Field. A geometrical algorithm for finding the solution of the implicit equation (36) in a certain fixed plane $x = x_0$ at the time $t = t_0$ is given below. Assuming that in (36) the function F is fixed, we determine the curve $\gamma = \{(y,z)\colon\ F(y,z) = 0\}$ specified by formula (36) for $\tau = 0$. To determine the angle ω at an arbitrary point $M = (y,z)$, it is necessary to construct a segment AM of length τ such that $A \in \gamma$. The required angle ω is specified by the direction of the segment AM (Fig. 2). The function ω is determined only at the points which are at a distance τ from the curve γ, and the remaining part of the plane Oyz is not the domain of definition of the function ω. The boundaries of the domain of definition of the function ω are τ-equidistant to γ. As x increases, the function τ varies as is specified by the solution of Eqs. (30)–(35). According to this, the domain of definition of the function ω in the planes $x = \text{const}$ also varies.

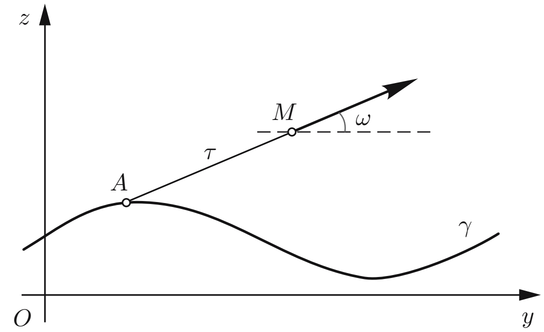

Fig. 2. Geometrical interpretation of the solutions $\omega = \omega(\tau(t, x), y, z)$ of the implicit equation (36).

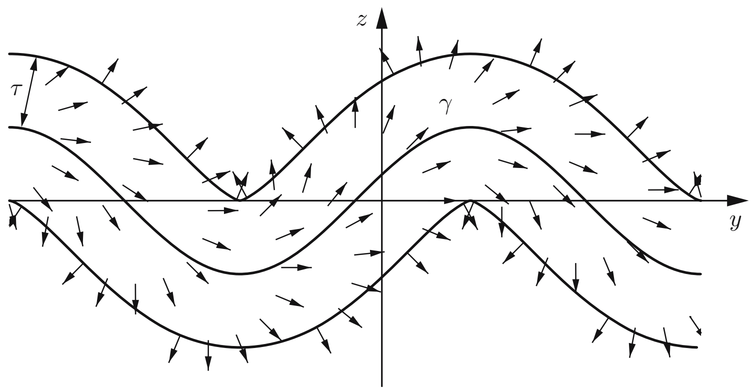

Fig. 3. Direction field defined by Eq. (36) for $F = z - \sin y$.

Thus, in each plane $x = \text{const}$, the domain of definition of the function ω [and, hence, of the entire solution (7)] is a strip of width 2τ with the curve γ as the middle line. On the boundaries of the domain of definition of ω, the corresponding direction field is orthogonal to the boundary. For the particular case $F = z - \sin y$, the direction field given by Eq. (36) is presented in Fig. 3.

Inside the domain of definition of the function ω is multiple-valued. Indeed, there can exist several segments AM of the required length with $A \in \gamma$, each of which defines the branch of the function ω. By choosing one of the branches, one can uniquely determine the function ω over the entire strip of definition. However, this function is not always continuous. In the case of large τ, where the boundary equidistants of the strip of definition have swallowtail singularities, discontinuities appear. In Fig. 4, it is shown that each branch of the function ω has a discontinuity line inside or on the boundary of the swallowtail. As the curve γ we choose a sinusoid, and the curve at the top of the figures is equidistant to γ, which is shifted for distance τ. The dashed curve shows a circle of radius τ with center at the point M. Each point of intersection of the circle with the curve ω corresponds to a branch of the function γ. If the point M is outside the swallowtail (Fig. 4a), at this point there are two branches of the function ω. As the point M moves toward the boundaries of the swallowtail, both branches of the function ω change continuously (Fig. 4b). Upon the intersection with the boundary, a new branch of the function ω appears (Fig. 4c). This branch is split into two branches inside the swallowtail (Fig. 4d). Once the point M reaches the right boundary, the two branches ω merge (Fig. 4e) and disappear when the point M leaves the swallowtail (Fig. 4f). It is easy to see that the branches of the function ω on the right of the swallowtail differ from the branches on the left of it. This means that, inside or on the boundary of the swallowtail, the function ω changes suddenly.

A swallowtail singularity does not appear for $\tau < \min\limits_{\boldsymbol{x} \in \gamma} R(\boldsymbol{x})$ [$R(\boldsymbol{x})$ is the curvature radius of the curve γ at the point $\boldsymbol{x}$]; therefore, singularities can be eliminated by choosing solutions with a small τ or by fixing the curve γ with a large curvature radius. The above singularity of the solution of the magnetohydrodynamic equations takes the solution out of the class (7). The resulting strong discontinuity cannot be interpreted as a shock wave because, on the discontinuity line, only the directions of the vector fields change and thermodynamic functions and

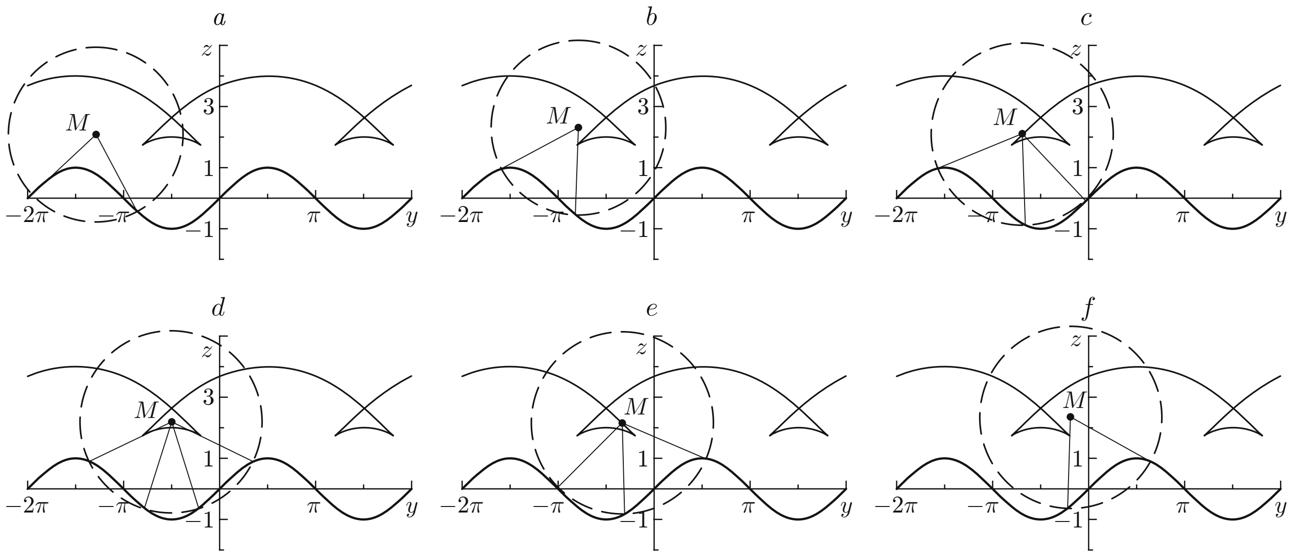

Fig. 4. Behavior of the function ω in the vicinity of the swallowtail singularity: branches of the function ω outside the swallowtail (a, b, and f), branches of the function ω on the boundaries of the swallowtail (c and e), and branches of the solution inside the swallowtail, dashed curve is a circle of radius τ with center at the point M (d).

the moduli of the velocity and magnetic-field vectors remain continuous. Using the rotational or Alfen discontinuity [14, 20], which is characteristic of ideal magnetohydrodynamics, also does not allow one to explain the occurrence of the discontinuity because the velocity and magnetic-field vectors rotate in a plane which is not tangent to the discontinuity plane.

From a physical point of view, the occurrence of singularities on the boundary of the domain of definition of the solution implies that the magnetic lines issuing from different points in the initial plane $x = \text{const}$ intersect each other. This occurs if the function τ increases along the magnetic line in such a manner that the τ-equidistants to γ become nonsmooth. In the vicinity of the point of intersection, the solution leaves the class described. The corresponding motion of continuous media should be described by the general equations of three-dimensional motion or by a more general model, for example, with magnetic and kinematic viscosities taken into account, as in problems of magnetic field line reconnection [21]. This nonlinear process is specific to the solution constructed and cannot be implemented in the classical one-dimensional motion, in which all magnetic lines are parallel.

3. CASE $\boldsymbol{H} = 0$

3.1. Equations of the Submodel. From the point of view of mechanics, the case $h = 0$ means that, in the projection onto the plane $x = \text{const}$, the vector fields $\boldsymbol{u}$ and $\boldsymbol{H}$ are incompressible, i.e., they have zero divergence. The noninvariant function ω for this case is found by using a different algorithm.

Thus, for $h = 0$, the integral (29) is absent; instead of it Eqs. (11) and (25) lead to

$$H = H_0 = \text{const}.$$

Thus, the equations of the invariant subsystem reduce to the system

$$\begin{gathered} \tilde{D}\rho + \rho U_x = 0, \qquad \tilde{D}U + \rho^{-1}p_x + \rho^{-1}NN_x = 0, \\ \tilde{D}V - \rho^{-1}H_0N_x = 0, \qquad \tilde{D}p + A(p,\rho)U_x = 0, \qquad \tilde{D}N + NU_x - H_0V_x = 0. \end{gathered} \tag{37}$$

System (37) serves to determine the unknown functions U, V, N, p, and ρ. The noninvariant function ω is determined from Eqs. (9), (27), and (28). To find its solution for $N \neq 0$ and $\rho V^2 - N^2 \neq 0$, we seek a dependence $\omega = \omega(t, x, y, z)$ in the implicit form $\Phi(t, x, y, z, \omega) = 0$, $\Phi_\omega \neq 0$. Then, the indicated system of equations is equivalent to the system

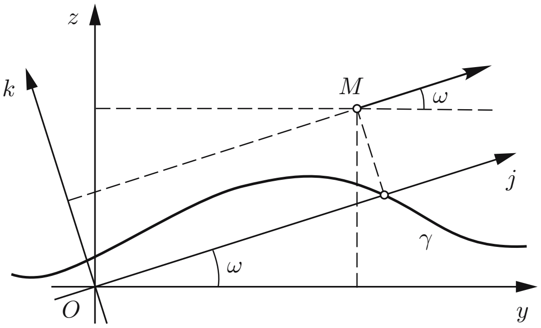

Fig. 5. Geometrical interpretation of the solution $\omega(t, x, j)$.

$$\Phi_k = 0, \qquad \Phi_t + U\Phi_x + V\Phi_j = 0, \qquad H_0\,\Phi_x + N\Phi_j = 0, \tag{38}$$

where Ojk is the Cartesian coordinate system rotated through the angle ω around the coordinate origin:

$$j = y\cos\omega + z\sin\omega, \qquad k = -y\sin\omega + z\cos\omega.$$

Integrals of system (38) are ω and $j - \varphi(t, x)$, where the function $\varphi(t, x)$ satisfies the overdetermined system

$$\varphi_t + U\varphi_x = V, \qquad H_0\,\varphi_x = N. \tag{39}$$

The compatibility condition for Eqs. (39) is the last equation of system (37). The differential 1-form

$$H_0\,d\varphi = (H_0 V - NU)\,dt + N\,dx$$

is closed, and, hence, the function φ is found by integration in the form

$$\varphi(t, x) = \int\limits_{(t_0, x_0)}^{(t, x)} d\varphi.$$

We note that the function φ is determined by its value at the initial point $\varphi(t_0, x_0)$. In the case considered, the noninvariant function ω is found in implicit form

$$j = f(\omega) + \varphi(t, x) \tag{40}$$

with an arbitrary smooth function f. The following theorem is valid.

Theorem 3. *In the case $h = 0$, the invariant functions are determined by solving system* (37), (39). *The noninvariant function ω is determined from the implicit equation* (40).

3.2. Construction and Properties of the Direction Field. Let us consider a geometrical interpretation of the solution $\omega(t, x, j)$ of the implicit finite equation (40). We fix the plane $x = x_0$ and the time $t = t_0$. To simplify the calculations, we set $\varphi(t_0, x_0) = 0$. It is also assumed that, at a certain point $M = (y, z)$ of the plane $x = x_0$, the quantity ω satisfying the implicit dependences (40) is specified. We consider the Cartesian coordinate system Ojk rotated counterclockwise through the angle ω around the system Oyz (Fig. 5). By the construction, the j-coordinate of the point M and the angle ω are linked by the relation $j = f(\omega)$, which is valid for all points of the plane with the same coordinate j and an arbitrary coordinate k.

The point with the coordinate $k = 0$ will be called the base point for the chosen values of j and ω that satisfy Eq. (40). The geometrical place of all base points for various j and ω will be called the base curve γ. In the plane Oyz, the base curve γ is defined in the polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$ by the equation $r = f(\theta)$. Since the quantity j can have an arbitrary sign, negative values of the polar coordinate r are also admitted in the construction of the curve γ.

Based on the above geometrical interpretation, it is possible to propose an algorithm for constructing the direction field determined by the angle ω of deviation from the positive Oy direction. The angle ω is found by solving the implicit equation (40). Under the assumption that the function f in Eq. (40) is specified, the base curve γ can be constructed by the formula $r = f(\theta)$ in the polar coordinates in the plane Oyz. To find the angle ω at the point $M = (y, z)$ in the plane $x = x_0$, it is necessary to construct a circle S_M of diameter OM and to find the points of intersection A_i of the circle S_M with the curve γ (Fig. 6). For each point A_i, the direction of the vector

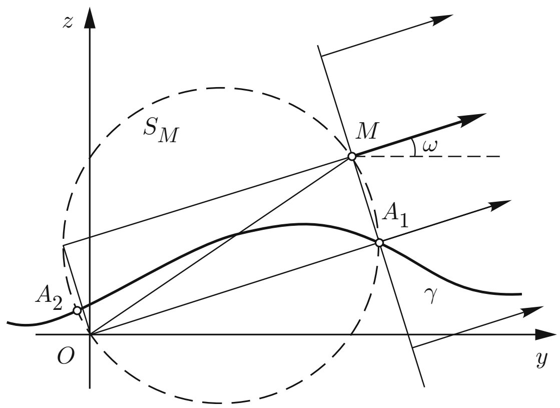

Fig. 6

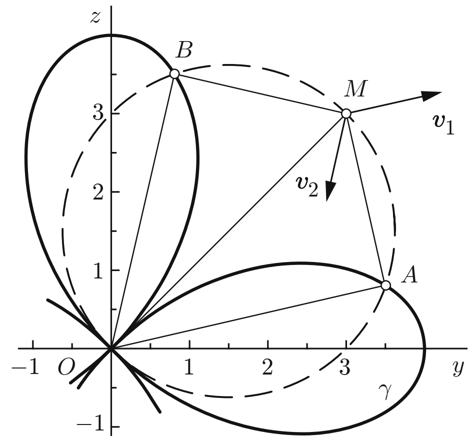

Fig. 7

Fig. 6. Geometry of the vector field corresponding to the point of intersection A_1.

Fig. 7. Geometry of the vector field for negative value of the function f.

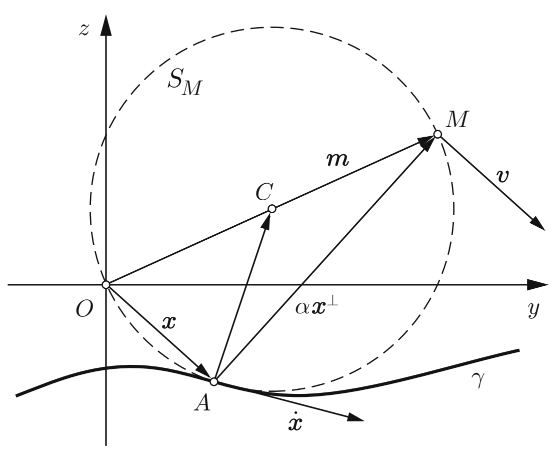

Fig. 8. Construction of the boundary of the domain of definition of the function ω.

field at the point M coincides with the direction of the segment OA_i. At all points of the straight line through the segment A_iM, the vector field has the same direction as that at the point M.

As noted above, the function f can take both negative and positive values. This implies that, in the construction of the curve γ, the coordinate r can also take negative values. The negative values of f correspond to the negative values of j. Thus, if the point of intersection of the auxiliary circle and the curve γ lies on the part of the curve that corresponds to the negative function f, the corresponding vector field should have the opposite direction. In Fig. 7, the curve γ is defined by the equation $r = \cos 2\theta$. The circle constructed on the diameter OM with the point $M = (3, 3)$ has two points of intersection with the curve γ (A and B). The point A lies on the part of the curve that corresponds to the positive value of the function f; therefore, it corresponds to the vector $\boldsymbol{v}_1$ codirected with the segment OA. The point B lies on the negative part of the arch γ; therefore, it corresponds to the vector $\boldsymbol{v}_2$ directed opposite to the segment OB.

Let us calculate the boundaries of the domain of definition of the solution $\omega = \omega(t, r, j)$ given by formula (40). We assume that the curve γ is specified. The point M is on the boundary of the domain of existence of the solution if the circle S_M constructed on the diameter OM is in contact to the curve γ at a certain point A (Fig. 8). We denote the radius-vector of the point M by $\boldsymbol{m}$ and assume that the curve γ is given in the parametric form

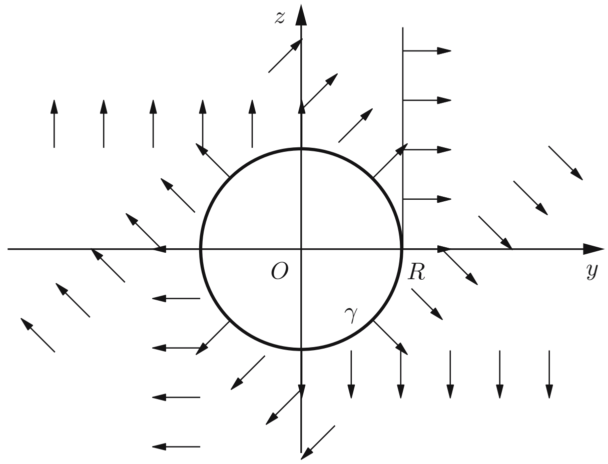

Fig. 9. Vector field defined by the curve $y^2 + z^2 = R^2$.

$\boldsymbol{x} = \boldsymbol{x}(s)$ with the parameter $s \in \Delta \subset \mathbb{R}$. It is obvious that $\boldsymbol{m} = \boldsymbol{x} + \alpha \boldsymbol{x}^\perp$, where $\boldsymbol{x}^\perp \cdot \boldsymbol{x} = 0$. In addition, from the condition of contact between the circle and the curve γ, it follows that $(\boldsymbol{m}/2 - \boldsymbol{x}) \cdot \dot{\boldsymbol{x}} = 0$ (the point above the symbol denotes differentiation with respect to s). Substitution of the expression for $\boldsymbol{m}$ from the first equality into the second yields $(\alpha \boldsymbol{x}^\perp/2 - \boldsymbol{x}/2) \cdot \dot{\boldsymbol{x}} = 0$. From this, we obtain

$$\alpha = \frac{\boldsymbol{x} \cdot \dot{\boldsymbol{x}}}{\boldsymbol{x}^\perp \cdot \dot{\boldsymbol{x}}}.$$

Thus, the boundary of the domain of definition of the function ω has the parametrization

$$\boldsymbol{m} = \boldsymbol{x} + \frac{\boldsymbol{x} \cdot \dot{\boldsymbol{x}}}{\boldsymbol{x}^\perp \cdot \dot{\boldsymbol{x}}}\, \boldsymbol{x}^\perp, \qquad \boldsymbol{x} = \boldsymbol{x}(s), \qquad s \in \Delta \subset \mathbb{R},$$

and $\boldsymbol{m}$ does not depend on the choice of the sign $\boldsymbol{x}^\perp$. At the points of the boundary, the vector field has the direction $\boldsymbol{x}$ orthogonal to the boundary:

$$\dot{\boldsymbol{m}} \cdot \boldsymbol{x} = (\dot{\boldsymbol{x}} + \dot{\alpha} \boldsymbol{x}^\perp + \alpha \dot{\boldsymbol{x}}^\perp) \cdot \boldsymbol{x} = \dot{\boldsymbol{x}} \cdot \boldsymbol{x} + \frac{\boldsymbol{x} \cdot \dot{\boldsymbol{x}}}{\boldsymbol{x}^\perp \cdot \dot{\boldsymbol{x}}}\, \dot{\boldsymbol{x}}^\perp \cdot \boldsymbol{x} = 0.$$

Because the relation $\boldsymbol{x} \cdot \boldsymbol{x}^\perp = 0$ implies that $\dot{\boldsymbol{x}} \cdot \boldsymbol{x}^\perp = -\boldsymbol{x} \cdot \dot{\boldsymbol{x}}^\perp$, the last expression vanishes.

As an example, we choose a circle $y^2 + z^2 = R^2$ as the curve γ. The boundary of the domain of definition of this vector field coincides with the circle γ because any its point $\boldsymbol{x}$ obeys the equality $\boldsymbol{x} \cdot \dot{\boldsymbol{x}} = 0$. The required vector field corresponding to the flow from the circular source is shown in Fig. 9. In the limit for $R = 0$ we obtain the vector field corresponding to rotation around the coordinate origin.

CONCLUSIONS

For the equations of ideal magnetohydrodynamics, a partially invariant submodel generated by motion of a plane was constructed and analyzed. The submodel is described by a system of equations with two independent variables and a finite relation containing a functional arbitrariness. A geometrical algorithm for the solution of the finite relation for the noninvariant function was proposed. It was shown that it is not always possible to choose the branch of the noninvariant function that is unique and smooth over the entire domain of definition. For the quantities included in the submodel, the range of values for which the solution can have singularities was determined.

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